Last time:

Computed one-loop scattering amplitude in φ^4- theory with "cutoff" Λ (Pauli-Villars):

$$\mathcal{M} \overset{\Lambda \gg m}{=} -i\lambda + iC\lambda^2\left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right)\right] + \mathcal{O}(\lambda^3) \qquad (1)$$

where λ is a function of Λ such that $\mathcal{M}$ is Λ-independent!

The experimentalist measures a certain amplitude $\lambda_p(s_0, t_0, u_0)$ at a particular center-of-mass energy, scattering angle etc. determined by

$$s_0 = (k_1 + k_2)^2$$
$$t_0 = (k_1 - k_3)^2$$
$$u_0 = (k_1 - k_4)^2$$

$$\rightarrow \quad -i\lambda_p = -i\lambda + iC\lambda^2\left[\log\left(\frac{\Lambda^2}{s_0}\right) + \log\left(\frac{\Lambda^2}{t_0}\right) + \log\left(\frac{\Lambda^2}{u_0}\right)\right] + \mathcal{O}(\lambda^3) \qquad (2)$$

Let us denote the sum of logarithms in eqs. (1) and (2) by L and L_0:

$$\mathcal{M} = -i\lambda + iC\lambda^2 L^2 + \mathcal{O}(\lambda^3) \quad (3a) \qquad -i\lambda_p = -i\lambda + iC\lambda^2 L_0 + \mathcal{O}(\lambda^3) \quad (3b)$$

this is how λ and λ_p are related!

Question: How can we express $\mathcal{M}$ in terms of experimentally measured λ_p?

$\rightarrow$ rearrange (3b) to obtain:

$$-i\lambda = -i\lambda_p - iC\lambda^2 L_0 + \mathcal{O}(\lambda^3)$$
$$= -i\lambda_p - iC\lambda_p^2 L_0 + \mathcal{O}(\lambda^3) \qquad (4)$$

higher order terms are determined by plugging (4) into (3b) and solving for coefficients such that r.h.s $= -i\lambda_p$

$\rightarrow$ plugging (4) into (3a) gives:

$$\mathcal{M} = -i\lambda + iC\lambda^2 L + \mathcal{O}(\lambda^3)$$
$$= -i\lambda_p - iC\lambda_p^2 L_0 + iC\lambda_p^2 L + \mathcal{O}(\lambda_p^3) \qquad (5)$$

$\rightarrow$ we now see that $\mathcal{M}$ is a function of $L - L_0 = \left[\log(s_0/s) + \log(t_0/t) + \log(u_0/u)\right]$

i.e. $$\mathcal{M} = -i\lambda_p + iC\lambda_p^2\left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right)\right] + \mathcal{O}(\lambda_p^3)$$

Note:

1) In the literature λ_p is often denoted by λ_R and is called "renormalized coupling constant"

2) In the path integral formulation, the scattering amplitude $\mathcal{M}$ is obtained by evaluating the integral

$$\int \mathcal{D}\varphi \; \varphi(x_1)\,\varphi(x_2)\,\varphi(x_3)\,\varphi(x_4) \times e^{i\int d^dx\left(\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] + \frac{\lambda}{4!}\varphi^4\right)}$$

· The regularization used here corresponds to restricting ourselves, in the integral $\int \mathcal{D}\varphi$, to integrating over only those field configurations $\varphi(x)$ whose Fourier transform $\varphi(k)$ vanishes for $k \geq \Lambda$.

Dimensional regularization

An alternative way to regularize the scattering amplitude is called

"dimensional regularization"

procedure: When we reach

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - c^2 + i\varepsilon)^2},$$ we rotate to

Euclidean space and generalize to D dimensions:

$$I(D) = i \int \frac{d_E^D k}{(2\pi)^D} \frac{1}{(k^2+c^2)^2}$$

$$= i \left[\frac{2\pi^{D/2}}{\Gamma(D/2)} \right] \frac{1}{(2\pi)^D} \int_0^\infty dk\, k^{D-1} \frac{1}{(k^2+c^2)^2}$$

→ changing the integration variable to $k^2 + c^2 = c^2/x$, we find

$$\int_0^\infty dk\, k^{D-1} \frac{1}{(k^2+c^2)^2} = \frac{1}{2} c^{D-4} \int_0^1 dx\, (1-x)^{D/2-1} x^{1-D/2}$$

Using the definition of the beta-function,

$$\int_0^1 dx\, x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

the above integral becomes

$$i\int \frac{d_E^D K}{(2\pi)^D} \frac{1}{(K^2+c^2)^2} = i\frac{1}{(4\pi)^{D/2}}\Gamma\left(\frac{4-D}{2}\right)c^{D-4}$$

As $D \to 4$, the r.h.s becomes

$$i\frac{1}{(4\pi)^2}\left(\frac{2}{4-D} - \log c^2 + \log(4\pi) - \gamma + \mathcal{O}(D-4)\right)$$

where $\gamma = 0.577\cdots$ denotes the Euler–Mascheroni constant.

→ We see that $\log \Lambda^2$ in Pauli-Villars regularization has been replaced by $\frac{2}{4-D}$

→ when physical quantities (measureable quantities) are replaced by physical coupling constants, all such poles cancel!

§4.2 Renormalizable versus Nonrenormalizable

We saw that in φ^4-theory, when expressing the scalar-scalar scattering amplitude in terms of the "physical" coupling constant λ_P, the dependence on the cutoff Λ disappears!

Question: Was this just a coincidence?
For which theories is this possible?

$\longrightarrow$ "renormalizable" versus "nonrenormalizable"

Dimensional analysis

Let us use units where $\hbar = c = 1$

$\rightarrow$ length and time have inverse of the dimension of mass

action $S = \int d^4x \, \mathcal{L}$ appears in path integral as e^{iS} $\rightarrow$ must be dimensionless

$\to$ $\mathcal{L}$ has dimension $[m]^4$
mass (or energy)

we use notation

$[\mathcal{L}] = 4, \quad [x] = -1, \quad [\partial] = 1$

Now consider the scalar field theory

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] - \frac{\lambda}{4!}\varphi^4$$

demand $[(\partial\varphi)^2] = 4 \longrightarrow [\varphi] = 1$

$\longrightarrow [\lambda] = 0 \quad ([\lambda] + 4[\varphi] = 4)$

How about the fermion field ψ?

$$\mathcal{L} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + \cdots$$

$[\mathcal{L}] = 4 \longrightarrow [\psi] = \frac{3}{2}$

Looking at Yukawa interaction $f\varphi\bar{\psi}\psi$,

we see $[f] = 0$

In contrast, for $\mathcal{L} = G\bar{\psi}\psi\bar{\psi}\psi$

(theory of weak interaction), $[G] = -2$

since $-2 + 4\left(\frac{3}{2}\right) = 4$

From the Maxwell Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$,
we see $[A_\mu] = 1$

$$[e A_\mu \bar{\psi}\gamma^\mu \psi] = 4 \longrightarrow [e] = 0$$

Scattering amplitude blows up

Consider $\mathcal{L} = i\bar{\psi}\not{\partial}\psi + G\bar{\psi}\psi\bar{\psi}\psi$

$\longrightarrow$ calculating amplitude $\mathcal{M}$ for 4-fermi interaction, we find to lowest order: $\mathcal{M} \sim G$

for next order: $\mathcal{M} \sim G + cG^2$

result of one-loop computation

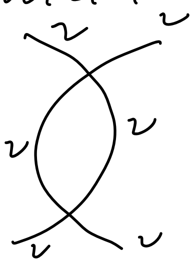
since $[G] = -2 \longrightarrow [c] = +2$

Assuming $m, k_i \ll \Lambda$, can set $m = k_i = 0$

$\longrightarrow$ For dimensional reasons, must have $c = \Lambda^2$

hence: $\mathcal{M} \sim G + \Lambda^2 G^2$

can also check this by comparing to Feynman diagram



$$\sim G^2 \int^{\Lambda} d^4p \left(\frac{1}{p}\right)\left(\frac{1}{\not{p}}\right) \sim G^2\Lambda^2$$

$\rightarrow$ becomes ∞ for $\Lambda = \infty$

"nonrenormalizable"

<u>Remark:</u>

The four-fermion interaction amplitude $\mathcal{M} \sim G + G^2 \Lambda^2$ signals that the theory is only valid below $\Lambda \sim (1/G)^{\frac{1}{2}}$ as at that energy perturbation theory breaks down!